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Ulam-Hyers stability and well-posedness of fixed point problems for α-λ-contractions on quasi b-metric spaces

Abdelbasset Felhi[1], Slah Sahmim[1] and Hassen Aydi[2,3*]

*Correspondence: hmaydi@uod.edu.sa
[2]Department of Mathematics, College of Education of Jubail, University of Dammam, P.O. Box 12020, Industrial Jubail, 31961, Saudi Arabia
[3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract

In this paper, we establish some fixed point results for α-λ-contractions in the class of quasi b-metric spaces. To illustrate the obtained results, we provide some examples and an application on a solution of an integral equation. We also study the stability of Ulam-Hyers and well-posedness of a fixed point problem. Our obtained results give an answer to an open problem of Kutbi and Sintunavarat (Abstr. Appl. Anal. 2014:268230, 2014).

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1 Introduction and preliminaries

By replacing the triangular inequality by a rectangular one, Czerwik [2] introduced a generalized metric space, named a b-metric space. Since then, several (common) fixed point papers have been obtained. For example, see [3–7]. Also, by lifting the symmetric condition, a quasi metric space generalizes the concept of a metric space. For some known fixed point results on these spaces, we refer to [8–12]. This paper deals with a combination of a b-metric and a quasi metric.

First, the definition of a quasi b-metric space is given as follows:

Definition 1.1 Let X be a nonempty and $s \geq 1$. Let $q : X \times X \to [0, \infty)$ be a function which satisfies:

(q1) $q(x, y) = 0$ if and only if $x = y$,

(q2) $q(x, y) \leq s[q(x, z) + q(z, y)]$.

Then q is called a quasi b-metric and the pair (X, q) is called a quasi b-metric space. The number s is called the coefficient of (X, q).

Remark 1.1 Any quasi metric space or any b-metric is a quasi b-metric space, but the converse is not true in general.

We state some examples of quasi b-metrics.

Example 1.1 Let $X = \{1, 2, 3\}$. Define the function $q : X \times X \to [0, \infty)$ by

$$q(n,m) = \begin{cases} \frac{1}{n^2} & \text{if } n > m, \\ 0 & \text{if } n = m, \\ 1 & \text{if } n < m, \end{cases}$$

for all $n, m \in X$, with $(n,m) \neq (1,2)$ and $q(1,2) = \frac{16}{9}$. Then (X, q) is a quasi b-metric space with coefficient $s = 2$. It is neither a b-metric space since $q(1,2) = \frac{16}{9} \neq q(2,1) = \frac{1}{4}$, nor a quasi metric space since $q(1,2) = \frac{16}{9} > \frac{10}{9} = q(1,3) + q(3,2)$.

Example 1.2 Let $X = \mathbb{R}$. Take the real numbers $p > 1$ and $a, b > 0$ such that $a \neq b$. Define the function $q : X \times X \to [0, \infty)$ by

$$q(x,y) = \bigl(\max\bigl\{a(x-y), b(y-x)\bigr\}\bigr)^p \quad \forall x, y \in X.$$

Then (X, q) is a quasi b-metric space with coefficient $s = 2^{p-1}$. It is neither a b-metric space since $q(1,0) = a^p \neq q(0,1) = b^p$, nor a quasi metric space since $q(1,-1) = (2a)^p > 2a^p = q(1,0) + q(0,-1)$.

Example 1.3 Let $X = \mathbb{R}$. Take the real numbers $p > 1$ and $a > 0$. Define the function $q : X \times X \to [0, \infty)$ by

$$q(x,y) = \begin{cases} (x-y)^p & \text{if } x \geq y; \\ (y-x+a)^p & \text{if } x < y. \end{cases}$$

Then (X, q) is a quasi b-metric space with coefficient $s = 2^{p-1}$. It is neither a b-metric space since $q(1,0) = 1 \neq q(0,1) = (1+a)^p$, nor a quasi metric space since $q(1,-1) = 2^p > 2 = q(1,0) + q(0,-1)$.

Some topological aspects of a quasi b-metric space are as follows.

Definition 1.2 Let (X, q) be a quasi b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0. \tag{1}$$

Remark 1.2 In a quasi b-metric space, the limit for a convergent sequence is unique. If $x_n \to u$, we have (in general) $\lim_{n\to\infty} q(x_n, y) \neq q(u, y)$ for all $y \in X$. We only mention that

$$\frac{1}{s} q(u,y) \leq \limsup_{n\to\infty} q(x_n, y) \leq s q(u,y).$$

Definition 1.3 Let (X, q) be a quasi b-metric space. A sequence $\{x_n\}$ in X is said left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_k) < \varepsilon$ for all $n \geq k > N$.

Definition 1.4 Let (X, q) be a quasi b-metric space. A sequence $\{x_n\}$ in X is said right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N_\varepsilon$ such that $q(x_n, x_k) < \varepsilon$ for all $k \geq n > N$.

Definition 1.5 Let (X, q) be a quasi b-metric space. A sequence $\{x_n\}$ in X is said Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N_\varepsilon$ such that $q(x_n, x_k) < \varepsilon$ for all $k, n > N$.

Remark 1.3 A sequence $\{x_n\}$ in a quasi b-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 1.6 Let (X, q) be a quasi b-metric space. We say that:
(1) (X, q) is left-complete if and only if each left-Cauchy sequence in X is convergent.
(2) (X, q) is right-complete if and only if each right-Cauchy sequence in X is convergent.
(3) (X, q) is complete if and only if each Cauchy sequence in X is convergent.

Lemma 1.1 *Let (X, q) be a quasi b-metric space and $T : X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then, for all sequence $\{x_n\}$ in X such that $x_n \to u$, we have $Tx_n \to Tu$, that is,*

$$\lim_{n\to\infty} q(Tx_n, Tu) = \lim_{n\to\infty} q(Tu, Tx_n) = 0.$$

In 2012, Samet *et al.* [13] introduced the notion of α-admissible maps.

Definition 1.7 [13] For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be given mappings. T is said α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1. \tag{2}$$

Using and generalizing the above concept, many authors established some (common) fixed point results. We may cite [14–18].

Very recently, Kutbi and Sintunavarat [1] introduced a new class of contractive mappings known as α-λ-contractions.

Definition 1.8 Let (X, d) be a metric space and $f : X \to X$ be a given mapping. We say that f is an α-λ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\lambda : X \to [0, 1)$ for which $\lambda(f(x)) \leq \lambda(x)$ for all $x \in X$, such that

$$\alpha(x, y) d\bigl(f(x), f(y)\bigr) \leq \lambda(x) d(x, y), \tag{3}$$

for all $x, y \in X$.

Starting from a question of Ulam [19] in 1940, the stability problem of functional equations concerns the stability of group homomorphisms. In 1941, Hyers [20] presented a partial answer for a question of Ulam in the case of Banach spaces. The above type of stability is known as Ulam-Hyers stability. Since then, many researchers extended and generalized the notion of the Ulam-Hyers stability for fixed point problems. For example, see [21–23].

Now, we introduce the concept of an α-λ-contractive mapping in the setting of quasi b-metric spaces.

Definition 1.9 Let (X,q) be a quasi b-metric space and $T:X\to X$ be a given mapping. We say that T is an α-λ-contraction if there exist $\alpha:X\times X\to[0,\infty)$ and $\lambda:X\to[0,\frac{1}{s})$ satisfying $\lambda(Tx)\le\lambda(x)$ for all $x\in X$, such that

$$\alpha(x,y)q(Tx,Ty)\le\lambda(x)q(x,y), \tag{4}$$

for all $x,y\in X$.

The following examples illustrate Definition 1.9.

Example 1.4 Going back to Example 1.1 where $X=\{1,2,3\}$ is endowed with the quasi b-metric $q:X\times X\to[0,\infty)$ defined by

$$q(n,m)=\begin{cases}\frac{1}{n^2} & \text{if } n>m,\\ 0 & \text{if } n=m,\\ 1 & \text{if } n<m,\end{cases}$$

for all $n,m\in X$ with $(n,m)\neq(1,2)$ and $q(1,2)=\frac{16}{9}$.

Define $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$ by

$$T1=3,\qquad T2=2,\qquad T3=1\quad\text{and}$$

$$\alpha(n,m)=\begin{cases}1 & \text{if } (n,m)\in X\times X-\{(2,1),(3,1),(3,2)\},\\ 0 & \text{otherwise.}\end{cases}$$

Since $q(T2,T1)=q(2,3)=1>\frac{1}{4}=q(2,1)$, T is not a Banach contraction on X. Now, we show that T is an α-λ-contraction, where $\lambda:X\to[0,\frac{1}{2})$ is defined by

$$\lambda(1)=\lambda(3)=\frac{1}{9}\quad\text{and}\quad\lambda(2)=\frac{1}{4}.$$

To this aim, we distinguish the following cases:

Case 1: If $n=1$, $m=2$, then we have

$$\alpha(1,2)q(T1,T2)=q(T1,T2)=q(3,2)=\frac{1}{9}\le\frac{1}{9}\times\frac{16}{9}=\lambda(1)q(1,2).$$

Case 2: If $n=1$, $m=3$, then we get

$$\alpha(1,3)q(T1,T3)=q(T1,T3)=q(3,1)=\frac{1}{9}\le\frac{1}{9}\times 1=\lambda(1)q(1,3).$$

Case 3: If $n=2$, $m=3$, then we get

$$\alpha(2,3)q(T2,T3)=q(T2,T3)=q(2,1)=\frac{1}{4}=\frac{1}{4}\times 1=\lambda(2)q(2,3).$$

Moreover, (4) is verified for all $n = m$ and for all $n, m \in X$ such that $\alpha(n,m) = 0$. Since $\lambda(T1) = \lambda(3) = \lambda(1)$, $\lambda(T2) = \lambda(2)$, and $\lambda(T3) = \lambda(1) = \lambda(3)$, the mapping T is an α-λ-contraction.

Example 1.5 Let $X = \{0,1\} \cup [2,\infty)$. Consider the mapping $q : X \times X \to [0,\infty)$ defined by

$$q(x,y) = \begin{cases} 4 & \text{if } x > y, \\ 0 & \text{if } x = y, \\ 1 & \text{if } x < y, \end{cases}$$

for all $x, y \in X$ with $(x,y) \neq (0,1)$ and $q(0,1) = 9$. We mention that (X,q) is a quasi b-metric space with $s = 2$. Note that q is not a quasi metric since $q(0,1) = 9 > 5 = q(0,2) + q(2,1)$.

Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} 2 + \frac{1}{x} & \text{if } x \geq 2, \\ 3 & \text{if } x \in \{0,1\}, \end{cases} \quad \text{and} \quad \alpha(x,y) = \begin{cases} 1 & \text{if } x > y \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$q(T1, T2) = q\left(3, \frac{5}{2}\right) = 4 > 1 = q(1,2),$$

that is, T is not a Banach contraction on X. Now, we show that T is an α-λ-contraction where $\lambda : X \to [0, \frac{1}{2})$ is defined by $\lambda(x) = \frac{1}{3}$ for all $x \in X$. To this aim, we distinguish the following cases:

Case 1: If $x, y \in X$ such that $\alpha(x,y) = 1$, then we have $x > y \geq 2$. It follows that

$$\alpha(x,y)q(Tx,Ty) = q(Tx,Ty) = q\left(2 + \frac{1}{x}, 2 + \frac{1}{y}\right) = 1 = \frac{1}{4} \times 4 = \lambda(x)q(x,y).$$

Case 2: If $(x,y) \in X$ such that $\alpha(x,y) = 0$, then (4) is verified.

Thus, (4) is satisfied and since $\lambda(Tx) = \lambda(x)$ for all $x \in X$, so the mapping T is an α-λ-contraction.

In this paper, we are interested in Ulam-Hyers stability and the well-posedness of the fixed point problem concerning α-λ-contraction mappings in the setting of quasi b-metric spaces. Our results are proper extensions and generalizations of results of Kutbi and Sintunavarat [1] on quasi b-metric spaces. Some examples and an application are also considered.

2 Auxiliary results

We have the following useful lemmas.

Lemma 2.1 *Let $X = \mathbb{R}$ and $p > 1$ be a real number. Consider the function $q : X \times X \to [0,\infty)$ by*

$$q(x,y) = \left(\max\{a(x-y), b(y-x)\}\right)^p \quad \forall x, y \in X, \tag{5}$$

where a and b are positive reals such that $a \neq b$. Then there exist two positive constants c and d such that

$$c|x-y|^p \leq q(x,y) \leq d|x-y|^p, \tag{6}$$

for all $x, y \in X$.

Proof Without loss of generality, we suppose that $a < b$. To this aim, we distinguish the following cases:

Case 1: If $x, y \in X$ such that $x > y$, then we have

$$q(x,y) = a^p(x-y)^p = a^p|x-y|^p \leq b^p|x-y|^p,$$

that is,

$$a^p|x-y|^p \leq q(x,y) \leq b^p|x-y|^p.$$

Case 2: If $x, y \in X$ such that $x \leq y$, then we get

$$q(x,y) = b^p(y-x)^p = b^p|x-y|^p \geq a^p|x-y|^p,$$

that is,

$$a^p|x-y|^p \leq q(x,y) \leq b^p|x-y|^p.$$

Consequently, we obtain (6), with $c = a^p$ and $d = b^p$. □

Lemma 2.2 *Let $X = \mathbb{R}$ be endowed with quasi b-metric q given by* (5). *Take $T : X \to X$. We have*

$$T \text{ is continuous on } (X,q) \quad \Longleftrightarrow \quad T \text{ is continuous on } \bigl(X, |\cdot|\bigr),$$

where $|\cdot|$ is the standard metric on X.

Proof Assume that T is continuous on $(X, |\cdot|)$. Consider $\{x_n\}$ in X such that $x_n \to x$ in (X,q). Then

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

By (6), we get $x_n \to x$ in $(X, |\cdot|)$. We deduce $Tx_n \to Tx$ in $(X, |\cdot|)$. Again, by (6)

$$\lim_{n\to\infty} q(Tx_n, Tx) = \lim_{n\to\infty} q(Tx, Tx_n) = 0,$$

that is, T is continuous on (X,q).

Similarly, if T is continuous on (X,q), then by (6), T is continuous on $(X, |\cdot|)$. □

3 Fixed point theorems

In this section, we shall state and prove our main results.

Theorem 3.1 *Let (X,q) be a complete quasi b-metric space and $T:X\to X$ be an α-λ-contraction. Suppose that*

(i) *T is an α-admissible mapping;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ and $\alpha(Tx_0,x_0)\geq 1$;*
(iii) *T is continuous on (X,q).*

Then T has a fixed point.

Proof By assumption (ii), there exists a point $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ and $\alpha(Tx_0,x_0)\geq 1$. Take $x_n=T^nx_0$ for all $n\geq 0$. From (i), we have by induction

$$\alpha(x_n,x_{n+1})\geq 1\quad\text{and}\quad\alpha(x_{n+1},x_n)\geq 1\quad\text{for all } n=0,1,\ldots. \tag{7}$$

Applying (4) with $x=x_0$ and $y=x_1$ and using (7), we get

$$q(x_1,x_2)=q(Tx_0,Tx_1)\leq\alpha(x_0,x_1)q(Tx_0,Tx_1)\leq\lambda(x_0)q(x_0,x_1).$$

We apply again (4) with $x=x_1$ and $y=x_2$ and using (7) together with the propriety of λ, we get

$$\begin{aligned}q(x_2,x_3)&=q(Tx_1,Tx_2)\leq\alpha(x_1,x_2)q(Tx_1,Tx_2)\\&\leq\lambda(x_1)q(x_1,Tx_2)\\&=\lambda(fx_0)q(x_1,x_2)\leq\lambda(x_0)q(x_1,x_2)\\&\leq\big[\lambda(x_0)\big]^2q(x_0,x_1).\end{aligned}$$

A similar argument leads to

$$q(x_n,x_{n+1})\leq\big[\lambda(x_0)\big]^nq(x_0,x_1). \tag{8}$$

The same procedure allows us to write

$$q(x_{n+1},x_n)\leq\big[\lambda(x_1)\big]^nq(x_1,x_0). \tag{9}$$

Since $\lambda(x_0)$ and $\lambda(x_1)$ are in $[0,1)$,

$$\lim_{n\to\infty}q(x_{n+1},x_n)=\lim_{n\to\infty}q(x_n,x_{n+1})=0. \tag{10}$$

We shall prove that $\{x_n\}$ is a Cauchy sequence in (X,q).

First, we claim that $\{x_n\}$ is a right-Cauchy sequence in the quasi b-metric space (X,q). Using (q2) and (8), we have for all $n,k\in\mathbb{N}$

$$\begin{aligned}q(x_n,x_{n+k})&\leq sq(x_n,x_{n+1})+s^2q(x_{n+1},x_{n+2})+\cdots+s^{k-1}q(x_{n+k-1},x_{n+k})\\&\leq\Big(s\big[\lambda(x_0)\big]^n+s^2\big[\lambda(x_0)\big]^{n+1}+\cdots+s^{k-1}\big[\lambda(x_0)\big]^{n+k-1}\Big)\big[q(x_0,x_1)\big]\end{aligned}$$

$$\begin{aligned}&\leq \sum_{i=n}^{n+k-1} s^i\big[\lambda(x_0)\big]^i\big[q(x_0,x_1)\big]\\&\leq \sum_{i=n}^{\infty} s^i\big[\lambda(x_0)\big]^i\big[q(x_0,x_1)\big].\end{aligned} \tag{11}$$

Since $s\lambda(x_0) < 1$,

$$q(x_n, x_{n+k}) \to 0 \quad \text{as } n \to \infty, \text{ for all } k. \tag{12}$$

It follows that $\{x_n\}$ is a right-Cauchy sequence in the quasi b-metric space (X,q). Similarly, using (9), we see that $\{x_n\}$ is a left-Cauchy sequence in the quasi b-metric space (X,q). We deduce that $\{x_n\}$ is a Cauchy sequence in the quasi b-metric space (X,q).

Since (X,q) is complete, the sequence $\{x_n\}$ converges to some $u \in X$, that is,

$$\lim_{n\to\infty} q(x_n,u) = \lim_{n\to\infty} q(u,x_n) = 0.$$

The continuity of T yields

$$\lim_{n\to\infty} q(Tx_n, Tu) = \lim_{n\to\infty} q(x_{n+1}, Tu) = 0. \tag{13}$$

By uniqueness of the limit, we get $Tu = u$. Therefore, u is a fixed point of T. □

Using the same techniques we obtain the following result.

Theorem 3.2 *Let (X,q) be a complete b-metric space and $T : X \to X$ be an α-λ-contraction. Suppose that*

(i) *T is an α-admissible mapping*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous on (X,q)*.

Then T has a fixed point.

Considering $s = 1$ in Theorem 3.1 (resp. Theorem 3.2), we have

Corollary 3.1 *Let (X,q) be a complete quasi metric space and $T : X \to X$ be an α-λ-contraction.*

Suppose that:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *T is continuous on (X,q)*.

Then there exists $u \in X$ such that $u = Tu$.

Corollary 3.2 (Theorem 10, [1]) *Let (X,d) be a complete metric space and $T : X \to X$ be an α-λ-contraction satisfying the following conditions*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous on (X,d)*.

Then there exists $u \in X$ such that $u = Tu$.

We may replace the continuity hypothesis of T in Theorem 3.1 (resp. Theorem 3.2) by one of the following hypotheses:

(H) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ and $\alpha(x_{n+1},x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$, for all k.

(R) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$, for all k.

Theorem 3.3 *Let (X,q) be a complete quasi b-metric space and $T : X \to X$ be an α-λ-contraction. Suppose that*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(Tx_0,x_0) \geq 1$*;
(iii) (H) *holds*.

Then there exists $u \in X$ such that $u = Tu$.

Proof Following the proof of Theorem 3.1, the sequence $\{x_n\}$ is Cauchy and converges to some $u \in X$ in (X,q). Remember that (7) holds, so from condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},u) \geq 1$, for all k. We shall show that $u = Tu$.

We have, for all $k \geq 0$,

$$q(u,Tu) \leq sq(u,x_{n(k)+1}) + sq(x_{n(k)+1},Tu). \tag{14}$$

Taking $x = x_{n(k)}$ and $y = u$ in (4), we obtain

$$\begin{aligned} q(x_{n(k)+1},Tu) &= q(Tx_{n(k)},Tu) \leq \alpha(x_{n(k)},u)q(Tx_{n(k)},Tu) \\ &\leq \lambda(x_{n(k)})q(x_{n(k)},u) \leq \lambda(x_0)q(x_{n(k)},u) < \frac{1}{s}q(x_{n(k)},u). \end{aligned}$$

Then we get for all $k \geq 0$

$$q(u,Tu) \leq sq(u,x_{n(k)+1}) + q(x_{n(k)},u). \tag{15}$$

Letting $k \to \infty$ in (15), we have

$$q(u,Tu) \leq 0.$$

This yields $Tu = u$. This completes the proof. □

We also state the following result. Its proof is very immediate.

Theorem 3.4 *Let (X,q) be a complete b-metric space and $T : X \to X$ be an α-λ-contraction. Suppose that*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$*;
(iii) (R) *holds*.

Then there exists $u \in X$ such that $u = Tu$.

Considering $s = 1$ in Theorem 3.3 (resp. Theorem 3.4), we have

Corollary 3.3 *Let (X,q) be a complete quasi metric space and $T : X \to X$ be an α-λ-contraction.*

Suppose that:

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*

(iii) (H) *holds.*

Then there exists $u \in X$ such that $u = Tu$.

Corollary 3.4 (Theorem 12, [1]) *Let (X,d) be a complete metric space and $T : X \to X$ be an α-λ-contraction satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) (H) *holds.*

Then there exists $u \in X$ such that $u = Tu$.

We provide the following examples.

Example 3.1 Let $X = [0,\infty)$. Consider $q(x,y) = (\max\{(x-y), 2(y-x)\})^2$ for all $x,y \in X$. We mention that (X,q) is a complete quasi b-metric space with $s = 2$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \ln(1+\frac{x}{2}) & \text{if } x \in [0,1], \\ 2(x-1) + \ln\frac{3}{2} & \text{if } x > 1, \end{cases} \quad \text{and} \quad \alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Now, we show that T is an α-λ-contraction where $\lambda : X \to [0,\frac{1}{2})$ is defined by $\lambda(x) = \frac{1}{4}$ for all $x \in X$. To this aim, we distinguish the following cases:

Case 1: $x,y \in X$ such that $x \geq y$ and $\alpha(x,y) = 1$. We have

$$\begin{aligned} \alpha(x,y)q(Tx,Ty) &= q(Tx,Ty) = \left(\ln\left(1+\frac{x}{2}\right) - \ln\left(1+\frac{y}{2}\right)\right)^2 \\ &\leq \frac{1}{4}(x-y)^2 = \frac{1}{4}q(x,y) = \lambda(x)q(x,y). \end{aligned}$$

Case 2: $x,y \in X$ such that $x < y$ and $\alpha(x,y) = 1$. Similarly, we get

$$\alpha(x,y)q(Tx,Ty) \leq \lambda(x)q(x,y).$$

Hence, (4) is verified and, since $\lambda(Tx) = \lambda(x)$ for all $x \in X$, the mapping T is an α-λ-contraction.

Note that T is α-admissible. Since T is continuous on $(X,|\cdot|)$ where $|\cdot|$ is the standard metric on X, by Lemma 2.2, T is continuous on (X,q). We mention that $\alpha(1,T1) = \alpha(T1,1) = 1$ and so condition (ii) of Theorem 3.1 is verified. Hence, all hypotheses of Theorem 3.1 hold. Note that $u = 0$ and $v = 2 - \ln(\frac{3}{2})$ are the two fixed points of T.

Example 3.2 Let $X = [0,1]$. Consider $q(x,y) = (\max\{x-y, 2(y-x)\})^2$ for all $x, y \in X$. We mention that (X,q) is a quasi b-metric space with $s = 2$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \frac{x^2}{2\sqrt{2}} & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1, \end{cases} \quad \text{and} \quad \alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1), \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$q\left(T1, T\frac{1}{2}\right) = q\left(\frac{3}{4}, \frac{1}{8\sqrt{2}}\right) = \left(\frac{3}{4} - \frac{1}{8\sqrt{2}}\right)^2 > \frac{1}{4} = q\left(1, \frac{1}{2}\right),$$

that is, T is not a Banach contraction on X. Now, we show that T is an α-λ-contraction where $\lambda : X \to [0, \frac{1}{2})$ is defined by

$$\lambda(x) = \begin{cases} \frac{(x+1)^2}{8} & \text{if } 0 \le x < 1, \\ \frac{2}{5} & \text{if } x = 1. \end{cases}$$

First of all, we show that $\lambda(Tx) \le \lambda(x)$ for all $x \in X$. For $x = 1$, we have $\lambda(T1) = \lambda(1)$. Also, for $x \in [0,1)$, we have

$$\lambda(Tx) = \lambda\left(\frac{x^2}{2\sqrt{2}}\right) = \frac{1}{8}\left(\frac{x^2}{2\sqrt{2}} + 1\right)^2 \le \frac{1}{8}(x+1)^2 = \lambda(x).$$

Again, we show that (4) is verified. To this aim, we distinguish the following cases:

Case 1: If $x, y \in [0,1)$ such that $x \le y$, then we have

$$\begin{aligned} \alpha(x,y)q(Tx,Ty) &= q(Tx,Ty) = \frac{1}{2}\left(y^2 - x^2\right)^2 = \frac{1}{2}(y-x)^2(x+y)^2 = \frac{1}{8}(x+y)^2 q(x,y) \\ &\le \frac{1}{8}(x+1)^2 q(x,y) = \lambda(x)q(x,y). \end{aligned}$$

Case 2: If $x, y \in [0,1)$ such that $x > y$, then we obtain

$$\alpha(x,y)q(Tx,Ty) \le \lambda(x)q(x,y).$$

Case 3: If $(x,y) \notin [0,1)^2$, then we have $\alpha(x,y) = 0$, and so (4) is verified.

Thus, (4) is satisfied and the mapping T is an α-λ-contraction.

Note that T is α-admissible. By Lemma 2.2, T is not continuous on (X,q), then Theorem 3.1 is not applicable. Also, it is easy to see that $\alpha(\frac{1}{2}, T\frac{1}{2}) = \alpha(T\frac{1}{2}, \frac{1}{2}) = 1$, and so condition (ii) of Theorem 3.3 is verified. Now, we show that condition (H) holds. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ and $\alpha(x_{n+1}, x_{n+}) \ge 1$ for all n and $x_n \to u$ in (X,q). Then $\{x_n\} \subset [0, \frac{1}{2}]$ and $x_n \to u$ in $(X, |\cdot|)$. Thus, $u \in [0, \frac{1}{2}]$ and so $\alpha(x_n, u) = \alpha(u, x_n) = 1$ for all n.

Therefore, all hypotheses of Theorem 3.3 are satisfied. Here, $\{0,1\}$ is the set of fixed points of T.

To prove uniqueness of the fixed point given in Theorem 3.1 (resp. Theorem 3.2, Theorem 3.3, Theorem 3.4), we need to take one of the following additional hypotheses:

(U) For all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$, where $F(T)$ denotes the set of fixed points of T.
(V) For all $x, y \in F(T)$, there exists $z \in X$ such that $\min\{\alpha(x, z), \alpha(z, y)\} \geq 1$.

Theorem 3.5 *Adding condition* (U) *to the hypotheses of Theorem* 3.1 (*resp. Theorem* 3.2, *Theorem* 3.3, *Theorem* 3.4), *we see that u is the unique fixed point of* T.

Proof We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Tu$ and $v = Tv$ with $u \neq v$. By (4) and the fact that $\alpha(u, v) \geq 1$, we get

$$0 < q(u, v) \leq \alpha(u, v)q(u, v) = \alpha(u, v)q(Tu, Tv) \leq \lambda(u)q(u, v) < q(u, v),$$

which is a contradiction. Hence, $u = v$. □

Theorem 3.6 *Adding condition* (V) *to the hypotheses of Theorem* 3.1 (*resp. Theorem* 3.2, *Theorem* 3.3, *Theorem* 3.4), *we see that u is the unique fixed point of* T.

Proof Suppose that there exist u, v, two fixed points of T. By condition (V), there exists $z \in X$ such that $\min\{\alpha(u, z), \alpha(z, v)\} \geq 1$. Since T is α-admissible, it follows that

$$\min\{\alpha(u, T^n z), \alpha(T^n z, v)\} \geq 1, \quad \forall n = 0, 1, \dots. \tag{16}$$

We have

$$\begin{aligned} q(u, T^{n+1}z) &\leq \alpha(u, T^n z)q(u, T^{n+1}z) = \alpha(u, T^n z)q(Tu, T(T^n z)) \\ &\leq \lambda(u)q(u, T^n z), \quad \forall n = 0, 1, \dots. \end{aligned} \tag{17}$$

By induction, we obtain

$$q(u, T^n z) \leq [\lambda(u)]^n q(u, z), \quad \forall n = 0, 1, \dots. \tag{18}$$

A similar reasoning shows that

$$q(T^n z, v) \leq [\lambda(z)]^n q(z, v), \quad \forall n = 0, 1, \dots. \tag{19}$$

On the other side, we have

$$q(u, v) \leq sq(u, T^n z) + sq(T^n z, v), \quad \forall n = 0, 1, \dots, \tag{20}$$

which yields

$$q(u, v) \leq s[\lambda(u)]^n q(u, z) + s[\lambda(z)]^n q(z, v), \quad \forall n = 0, 1, \dots. \tag{21}$$

Passing to the limit as $n \to \infty$, we obtain

$$q(u, v) \leq 0, \tag{22}$$

and so $u = v$. □

The following examples illustrate Theorem 3.5.

Example 3.3 Let $X = \{0,1,2,3\}$. Consider the function $q : X \times X \to [0,\infty)$ defined by

$$q(n,m) = \begin{cases} 4 & \text{if } n > m, \\ 0 & \text{if } n = m, \\ 1 & \text{if } n < m, \end{cases}$$

for all $n,m \in X$ with $(n,m) \neq (0,1)$ and $q(0,1) = 9$. We mention that (X,q) is a complete quasi b-metric space with $s = 2$.

Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$T0 = 3, \qquad T1 = 0, \qquad T2 = T3 = 2 \quad \text{and} \quad \alpha(n,m) = \begin{cases} 1 & \text{if } n,m \in \{2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$q(T0,T2) = q(3,2) = 4 > 1 = q(0,2),$$

that is, T is not a Banach contraction on X. Now, we show that T is an α-λ-contraction where $\lambda : X \to [0,\frac{1}{2})$ is defined by $\lambda(n) = \frac{1}{4}$ for all $n \in X$. To this aim, we distinguish the following cases:

Case 1: If $n,m \in X$ such that $\alpha(n,m) = 1$, then $n,m \in \{2,3\}$. So

$$\alpha(n,m)q(Tn,Tm) = q(2,2) = 0 \leq \lambda(n)q(n,m).$$

Case 2: If $n,m \in X$ such that $\alpha(n,m) = 0$, then (4) is satisfied.

Thus, (4) holds and since $\lambda(Tn) = \lambda(n)$ for all $n \in X$, so the mapping T is an α-λ-contraction.

Note that T is α-admissible. In fact, let $n,m \in X$ such that $\alpha(n,m) \geq 1$, then $n,m \in \{2,3\}$, and so $\alpha(Tn,Tm) = \alpha(2,2) = 1$. Moreover, T is continuous on (X,q). In fact if $\{x_n\}$ is a sequence in X such that $x_n \to u$ in (X,q), it easy to see that there exists $N \in \mathbb{N}$ such that $x_n = u$ for all $n \geq N$ and so $Tx_n = Tu$ for all $n \geq N$. It follows that $\lim_{n\to\infty} q(Tx_n,Tu) = 0$, that is, T is continuous on (X,q). Also, since $\alpha(3,T3) = \alpha(3,2) = 1$, and $\alpha(T3,3) = \alpha(2,3) = 1$, condition (ii) of Theorem 3.3 is verified. Therefore, all hypotheses of Theorem 3.3 are satisfied. Here, 2 is the unique fixed point of T.

Example 3.4 Going back again to Example 3.2 where $X = [0,1]$ is endowed with the quasi b-metric $q(x,y) = (\max\{x-y, 2(y-x)\})^2$ for all $x,y \in X$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \frac{x^2}{2\sqrt{2}} & \text{if } 0 \leq x < 1, \\ \frac{3}{4} & \text{if } x = 1, \end{cases} \quad \text{and} \quad \alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,\frac{1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

We know that T is an α-λ-contraction where $\lambda : X \to [0, \frac{1}{2})$ is defined by

$$\lambda(x) = \begin{cases} \frac{(x+1)^2}{8} & \text{if } 0 \leq x < 1, \\ \frac{2}{5} & \text{if } x = 1. \end{cases}$$

Therefore, all hypotheses of Theorem 3.3 are satisfied. Here, 0 is the unique fixed point of T.

4 Fixed point results in quasi *b*-metric spaces endowed with a graph

Recently, Jachymski [24] introduced the concept of a G-contraction in the setting of metric spaces endowed with a graph. Using this notion, he proved some fixed point results. In this paragraph, we introduce a new class of contractive mappings in the setting of quasi b-metric spaces endowed with a graph. First, we recall some notations and definitions.

Let (X, q) be a quasi b-metric space and $\Delta = \{(x, x) : x \in X\}$ denote the diagonal of the cartesian product $X \times X$. Following [24], a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, *i.e.*, $\Delta \subset E(G)$. Also, we assume that G has no parallel edges and we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 4.1 [24] Let X be a nonempty set endowed with a graph G. We say that $T : X \to X$ weakly preserves edges of G if for all $x, y \in X$

$$(x, y) \in E(G) \quad \Rightarrow \quad (Tx, Ty) \in E(G).$$

Definition 4.2 Let (X, q) be a quasi b-metric space endowed with a graph G. We say that:

(1) (X, q) is G-complete if $\{x_n\}$ is a Cauchy sequence in X such that $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ for all n, then $\{x_n\}$ converges in (X, q).

(2) $T : X \to X$ is G-continuous if for each sequence $\{x_n\}$ such that $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ for all n and $x_n \to x$, then $Tx_n \to Tx$ in (X, q).

Remark 4.1 Let (X, q) be a quasi b-metric space endowed with a graph G.

(1) If (X, q) is a complete, then it is G-complete.

(2) If $T : X \to X$ is continuous on (X, q), then it is G-continuous.

We introduce the notion of a G-λ-contractive mapping in the class of quasi b-metric spaces endowed with a graph G.

Definition 4.3 Let (X, q) be a quasi b-metric space endowed with a graph G. A mapping $T : X \to X$ is said to be a G-λ-contraction if there exists a function $\lambda : X \to [0, \frac{1}{s})$ for which $\lambda(Tx) \leq \lambda(x)$ for all $x \in X$ such that

$$q(Tx, Ty) \leq \lambda(x) q(x, y), \tag{23}$$

for all $x, y \in X$ satisfying $(x, y) \in E(G)$.

We obtain the following results.

Theorem 4.1 *Let (X,q) be a quasi b-metric space endowed with a graph G and $T: X \to X$ be a G-λ-contraction. Suppose that*:

(i) *T weakly preserves edges of G*;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0), (Tx_0, x_0) \in E(G)$*;

(iii) *T is G-continuous on (X,q)*;

(iv) *(X,q) is G-complete*.

Then T has a fixed point.

Proof Define the function $\alpha: X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all conditions of Theorem 3.1 are satisfied and so T has a fixed point. □

Corollary 4.1 *Let (X,q) be a complete quasi b-metric space endowed with a graph G and $T: X \to X$ be a G-λ-contraction. Suppose that*:

(i) *T weakly preserves edges of G*;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0), (Tx_0, x_0) \in E(G)$*;

(iii) *T is G-continuous on (X,q)*.

Then T has a fixed point.

Corollary 4.2 *Let (X,q) be a complete quasi b-metric space endowed with a graph G and $T: X \to X$ be a G-λ-contraction. Suppose that*:

(i) *T weakly preserves edges of G*;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0), (Tx_0, x_0) \in E(G)$*;

(iii) *T is continuous on (X,q)*.

Then T has a fixed point.

Theorem 4.2 *Let (X,q) be a quasi b-metric space endowed with a graph G and $T: X \to X$ be a G-λ-contraction. Suppose that*:

(i) *T weakly preserves edges of G*;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0), (Tx_0, x_0) \in E(G)$*;

(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ for all n, then there exists $\{x_{n(k)}\}$ a subsequence of $\{x_n\}$ such that $(x_{n(k)}, u) \in E(G)$ for all k*;

(iv) *(X,q) is G-complete*.

Then T has a fixed point.

Corollary 4.3 *Let (X,q) be a complete quasi b-metric space endowed with a graph G and $T: X \to X$ be a G-λ-contraction. Suppose that*:

(i) *T weakly preserves edges of G*;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0), (Tx_0, x_0) \in E(G)$*;

(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ for all n, then there exists $\{x_{n(k)}\}$ a subsequence of $\{x_n\}$ such that $(x_{n(k)}, u) \in E(G)$ for all k.*

Then T has a fixed point.

5 Application

In this section, we apply Theorem 4.2 to the existence of a solution of an integral equation.

Let $X = C([a,b],\mathbb{R})$ be the set of real continuous functions defined on $[a,b]$. Consider the quasi b-metric $q_\infty : X \times X \to [0,\infty)$ given as follows:

$$q_\infty(x,y) = \sup_{t\in[a,b]} \big(\max\{x(t) - y(t), 2\big(y(t) - x(t)\big)\}\big)^2, \quad \forall x,y \in X.$$

We mention that (X,q) is a complete quasi b-metric space with $s = 2$. Also, suppose that X is endowed with a graph G. Consider the integral equation as follows:

$$x(t) = f(t) + \int_a^b K\big(t,s,x(s)\big)\,ds, \tag{24}$$

where $f : [a,b] \to \mathbb{R}$ and $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Let $T : X \to X$ be a mapping defined by

$$Tx(t) = f(t) + \int_a^b K\big(t,s,x(s)\big)\,ds. \tag{25}$$

It is clear that x is a solution of (24) if and only if x is a fixed point of T.

We have the following result.

Theorem 5.1 *Suppose that there exists $r \in [0, \frac{1}{\sqrt{2}})$ such that for all $t,s \in [a,b]$ we have*

$$0 \le K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big) \le \frac{r}{b-a}\big|x(s) - y(s)\big|, \tag{26}$$

for all $x,y \in X$ satisfying $(x,y) \in E(G)$.

Also, suppose that:

(i) *T weakly preserves edges of G*;

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0), (Tx_0, x_0) \in E(G)$*;

(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ for all n, then there exists $\{x_{n(k)}\}$ a subsequence of $\{x_n\}$ such that $(x_{n(k)}, u) \in E(G)$ for all k.*

Then the integral equation (24) *has a solution in $C([a,b],\mathbb{R})$.*

Proof Let $Tx(t) = f(t) + \int_a^b K(t,s,x(s))\,ds$. We shall show that it is a G-λ-contraction where $\lambda(x) = r^2$ for all $x \in X$.

Let $(x,y) \in E(G)$, then we get

$$\begin{aligned} 0 \le Tx(t) - Ty(t) &= \int_a^b \big[K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big)\big]\,ds \le \frac{r}{b-a}\int_a^b \big|x(s) - y(s)\big|\,ds \\ &\le \frac{r}{b-a}\left(\int_a^b ds\right)^{\frac12}\left(\int_a^b \big(x(s) - y(s)\big)^2\,ds\right)^{\frac12} \le r\big\|(x-y)^2\big\|_\infty^{\frac12}, \end{aligned}$$

where $\|(x-y)^2\|_\infty = \sup_{t\in[a,b]}(x(t) - y(t))^2$. It follows that

$$q_\infty(Tx,Ty) = \sup_{t\in[a,b]}\big(Tx(t) - Ty(t)\big)^2 \le r^2\big\|(x-y)^2\big\|_\infty \le r^2 q_\infty(x,y).$$

Hence, all conditions of Theorem 4.2 are satisfied and hence T has a fixed point in X. $\square$

6 Ulam-Hyers stability

Let (X, q) be a quasi b-metric space and $T : X \to X$ be a given mapping. Let us consider the fixed point equation

$$x = Tx \tag{27}$$

and the inequality (for $\varepsilon > 0$)

$$q(Ty, y) < \varepsilon. \tag{28}$$

We say that the fixed point problem (27) is Ulam-Hyers stable in the framework of a quasi b-metric space if there exists $c > 0$, such that for each $\varepsilon > 0$ and an ε-solution $v^* \in X$, that is, v^* satisfies the inequality (28), there exists a solution $u^* \in X$ of the fixed point equation (27) such that

$$q\bigl(u^*, v^*\bigr) < c\varepsilon. \tag{29}$$

Theorem 6.1 *Let (X, q) be a complete quasi b-metric space with coefficient s. Suppose that all the hypotheses of Theorem* 3.5 (*resp. Theorem* 3.6) *hold and $\alpha(w, z) \geq 1$ for all ε-solutions w, z, then the fixed point equation* (27) *is Ulam-Hyers stable*.

Proof By Theorem 3.5 (resp. Theorem 3.6), we have a unique $u \in X$ such that $u = Tu$, that is, $u \in X$ is a solution of the fixed point equation (27). Let $\varepsilon > 0$ and $v \in X$ be an ε-solution, that is,

$$q(Tv, v) \leq \varepsilon.$$

Since $q(u, Tu) = q(u, u) = 0 \leq \varepsilon$, u and v are ε-solutions. By hypothesis, we get $\alpha(u, v) \geq 1$ and so

$$\begin{aligned} q(u, v) = q(Tu, v) &\leq s\bigl[q(Tu, Tv) + q(Tv, v)\bigr] \leq s\alpha(u, v)q(Tu, Tv) + s\varepsilon \\ &\leq s\lambda(u)q(u, v) + s\varepsilon. \end{aligned}$$

We deduce

$$q(u, v) \leq \frac{s}{1 - s\lambda(u)}\varepsilon = c\varepsilon,$$

where $c = \frac{s}{1-s\lambda(u)} > 0$. Consequently, the fixed point problem of T is Ulam-Hyers stable. □

7 Well fixed point problem

Many mathematicians are interested in the concept of well-posedness of a fixed point problem. For instance, see [1, 25–27]. As in [9], we start to characterize the concept of the well-posedness in the context of quasi b-metric spaces as follows.

Definition 7.1 Let (X, q) be a quasi b-metric space and $T : T \to X$ be a given mapping. The fixed point problem (27) is said to be well posed if:

(1) T has a unique fixed point $u^* \in X$;
(2) for any sequence $\{x_n\} \subseteq X$ with $\lim_{n\to\infty} q(x_n, Tx_n) = \lim_{n\to\infty} q(Tx_n, x_n) = 0$, then we have $\lim_{n\to\infty} q(x_n, u^*) = \lim_{n\to\infty} q(u^*, x_n) = 0$.

In the following results, we need new conditions to ensure the well-posedness via α-admissibility:

(S_1) if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} q(x_n, Tx_n) = \lim_{n\to\infty} q(Tx_n, x_n) = 0$, then $\alpha(x_n, u^*) \geq 1$ and $\alpha(u^*, x_n) \geq 1$ for all n where u^* is a fixed point of T;
(S_2) if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} q(Tx_n, x_n) = 0$, then $\alpha(u^*, x_n) \geq 1$ for all n where u^* is a fixed point of T.

Theorem 7.1 *Let* (X, q) *be a complete quasi b-metric space with coefficient s and* $T : X \to X$ *be a given mapping. Suppose that all the hypotheses of Theorem* 3.5 (*resp. Theorem* 3.6) *hold.*

Also, suppose that:
(i) (S_1) *holds*;
(ii) *if* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} q(x_n, Tx_n) = \lim_{n\to\infty} q(Tx_n, x_n) = 0$, *then there exists* $N \in \mathbb{N}$ *such that* $\lambda(x_n) \leq \lambda(x_N)$, *for all* $n \geq N$.

Then the fixed point equation (27) *is well posed.*

Proof By Theorem 3.5 (resp. Theorem 3.6), we have a unique $u \in X$ such that $u = Tu$, that is, u is a solution of the fixed point equation (27). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} q(x_n, Tx_n) = \lim_{n\to\infty} q(Tx_n, x_n) = 0$. From condition ($S_1$), we have $\alpha(x_n, u) \geq 1$ and $\alpha(u, x_n) \geq 1$, for all n. Using (q2) and the fact that $\alpha(x_n, u) \geq 1$ in (4), one writes

$$\begin{aligned} q(x_n, u) &\leq sq(x_n, Tx_n) + sq(Tx_n, u) = sq(x_n, Tx_n) + sq(Tx_n, Tu) \\ &\leq sq(x_n, Tx_n) + s\alpha(x_n, u)q(Tx_n, Tu) \leq sq(x_n, Tx_n) + s\lambda(x_n)q(x_n, u). \end{aligned}$$

By condition (ii) of Theorem 7.1, we get

$$q(x_n, u) \leq sq(x_n, Tx_n) + s\lambda(x_N)q(x_n, u), \quad \forall n \geq N,$$

that is,

$$q(x_n, u) \leq \frac{s}{1 - s\lambda(x_N)} q(x_n, Tx_n), \quad \forall n \geq N.$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} q(x_n, u) = 0. \tag{30}$$

Again, using $\alpha(u, x_n) \geq 1$

$$\begin{aligned} q(u, x_n) &\leq sq(u, Tx_n) + sq(Tx_n, x_n) = sq(Tu, Tx_n) + sq(Tx_n, x_n) \\ &\leq s\alpha(u, x_n)q(Tu, Tx_n) + sq(Tx_n, x_n) \leq s\lambda(u)q(u, x_n) + sq(Tx_n, x_n). \end{aligned}$$

We deduce

$$q(u,x_n) \le \frac{s}{1 - s\lambda(u)} q(Tx_n, x_n).$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} q(u,x_n) = 0. \tag{31}$$

By (30) and (31), the fixed point problem (27) is well posed. □

Theorem 7.2 *Let* (X,q) *be a complete b-metric space with coefficient s and* $T : X \to X$ *be a given mapping. Suppose that all the hypotheses of Theorem* 3.5 (*resp. Theorem* 3.6) *hold. If* (S_2) *holds, then the fixed point equation* (27) *is well posed.*

Proof The proof is similar to that of Theorem 7.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Sciences, King Faisal University, Al-Hassa, Saudi Arabia. [2]Department of Mathematics, College of Education of Jubail, University of Dammam, P.O. Box 12020, Industrial Jubail, 31961, Saudi Arabia. [3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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